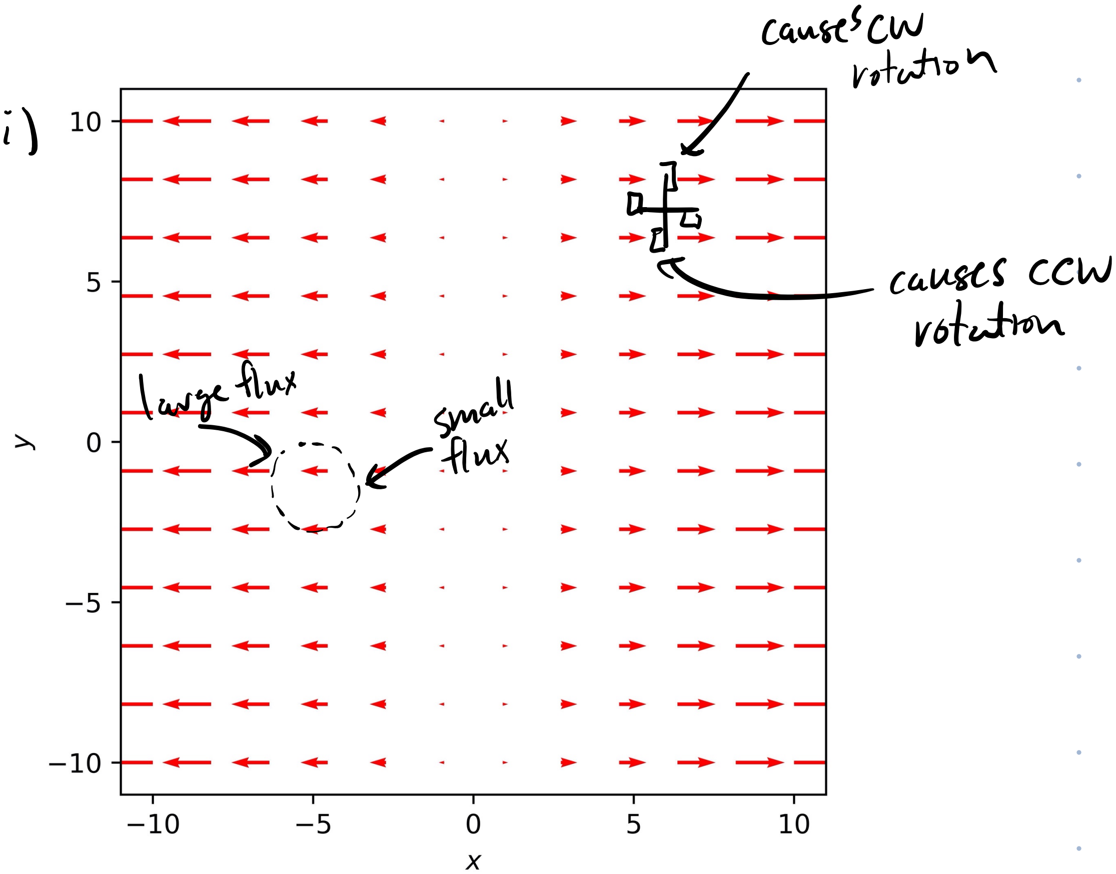
PHYS 301 - Tutorial #1 Sept. 10, 2024

Group problem solutions

1. (a) $\vec{V} = x\,\hat{x}$

(i) 

(ii) Place a small sphere in the vector field. $\vec{\nabla}\cdot\vec{V} = 0$ if net flux through sphere is zero.

In the sphere in the figure, have large flux exiting left side & small flux entering right side. Since net flux non-zero,

$\vec{\nabla}\cdot\vec{v} \neq 0.$

(iii) $$\vec{\nabla}\cdot\vec{V} = \frac{\partial V_x}{\partial x} + \cancel{\frac{\partial V_y}{\partial y}}^{\to 0} + \cancel{\frac{\partial V_z}{\partial z}}^{\to 0}$$

$$\boxed{\therefore \vec{\nabla}\cdot\vec{V} = \frac{\partial}{\partial x}(x) = 1.}$$

(iv) Place a small paddle wheel in the vector field. If paddle wheel would rotate, then $\vec{\nabla}\times\vec{V} \neq 0$

In this case, paddle wheel does <u>not</u> rotate b/c top & bottom influences cancel.

(v)

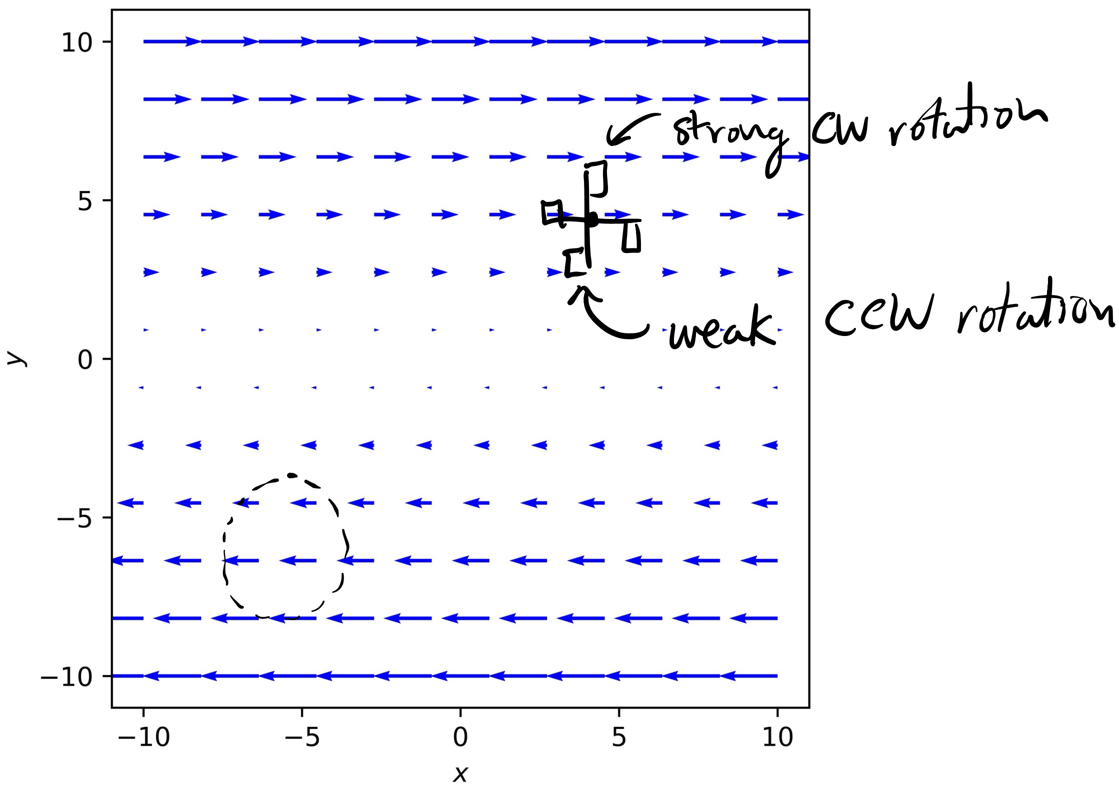
$$\vec{\nabla}\times\vec{V} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & 0 & 0 \end{vmatrix}$$

$$= \hat{x}(0-0) - \hat{y}\Big(0 - \underbrace{\frac{\partial x}{\partial z}}_{0}\Big) + \hat{k}\Big(0 - \underbrace{\frac{\partial x}{\partial y}}_{0}\Big) = \vec{0}.$$

$$\boxed{\therefore \vec{\nabla}\times\vec{V} = \vec{0}}$$

(b) $\vec{V} = y\hat{x}$

(i)



(ii) At all values of y, same flux enters & exits sphere. $\therefore\ \vec{\nabla}\cdot\vec{V} = 0$.

(iii) $$\vec{\nabla}\cdot\vec{V} = \frac{\partial V_x}{\partial x} + \cancelto{0}{\frac{\partial V_y}{\partial y}} + \cancelto{0}{\frac{\partial V_z}{\partial z}}$$

$$\boxed{\therefore\ \vec{\nabla}\cdot\vec{V} = \frac{\partial y}{\partial x} = 0.}$$

(iv) Paddle wheel rotates cw b/c of different magnitudes of $\vec{v}$ at top & bottom.

$\therefore \vec{\nabla} \times \vec{v} \neq 0.$

(v)

$$\vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & 0 & 0 \end{vmatrix}$$

$$= \hat{x}(0-0) - \hat{y}\Big(0 - \underbrace{\frac{\partial y}{\partial z}}_{0}\Big) + \hat{z}\Big(0 - \frac{\partial y}{\partial y}\Big)$$

$$\boxed{\therefore \vec{\nabla} \times \vec{v} = -\hat{z}}$$

2. $T = \dfrac{3x^2 y}{z}$

(a) $\vec{\nabla} T = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z} \right) \dfrac{3x^2 y}{z}$

$$= \frac{6xy}{z}\hat{x} + \frac{3x^2}{z}\hat{y} - \frac{3x^2 y}{z^2}$$

$$= \frac{3x^2 y}{z}\left(\frac{2}{x}\hat{x} + \frac{1}{y}\hat{y} - \frac{1}{z}\hat{z} \right)$$

$$\boxed{\therefore \vec{\nabla} T = T\left(\frac{2}{x}\hat{x} + \frac{1}{y}\hat{y} - \frac{1}{z}\hat{z} \right)}$$

(b) $\vec{\nabla} T$ points in dir'n of steepest ascent

$\therefore$ dir'n of steepest descent is parallel to $-\vec{\nabla} T$, or parallel to

$$\vec{v} = -\frac{2}{x}\hat{x} - \frac{1}{y}\hat{y} + \frac{1}{z}\hat{z}$$

@ $(1,1,1)$ $\vec{v} = -2\hat{x} - \hat{y} + \hat{z}$

To construct the required unit vector,

use $\hat{v} = \dfrac{\vec{v}}{v}$ where $v^2 = \vec{v} \cdot \vec{v}$

$$= (-2\hat{x} - \hat{y} + \hat{z}) \cdot (-2\hat{x} - \hat{y} + \hat{z})$$

$$= 4 + 1 + 1 = 6$$

$$\therefore v = \sqrt{6}$$

$$\boxed{\therefore \hat{v} = \frac{1}{\sqrt{6}}\left(-2\hat{x} - \hat{y} + \hat{z}\right)}$$

Check: If $\vec{v} \parallel -\vec{\nabla}T$, then expect $-\vec{\nabla}T \times \vec{v} = 0$

$$-\vec{\nabla}T \times \vec{v} = \frac{3}{\sqrt{6}} \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -2 & -1 & 1 \\ -2 & -1 & 1 \end{vmatrix}$$

$$= \sqrt{\frac{3}{2}} \left[\hat{x}(-1+1) - \hat{y}(-2+2) + \hat{z}(2-2) \right]$$

$$= \vec{0} \checkmark$$

(c) To find a vector $\perp$ to $\vec{\nabla} T$, require

$$\vec{V} \cdot \vec{\nabla} T = 0$$

@ $(1, 1, 1)$

$$\vec{V} \cdot \vec{\nabla} T = \left(V_x \hat{x} + V_y \hat{y} + V_z \hat{z} \right) \cdot 3 \left(2\hat{x} + \hat{y} - \hat{z} \right)$$

$$= 3 \left(2V_x + V_y - V_z \right) = 0$$

Take $V_x = 0$, $V_y = V_z = 1$.

$$\therefore \vec{V} = 3 (\hat{y} + \hat{z})$$

$$\therefore V^2 = \vec{V} \cdot \vec{V} = 9(1+1) = 18$$

$$\therefore V = \sqrt{18} = 3\sqrt{2}$$

$$\boxed{\therefore \hat{V} = \frac{\vec{V}}{V} = \frac{1}{\sqrt{2}} (\hat{y} + \hat{z})}$$

Check $\hat{v} \cdot \vec{\nabla} T = \frac{1}{\sqrt{2}}(\hat{y}+\hat{z}) \cdot 3(2\hat{x}+\hat{y}-\hat{z})$

$$= \frac{3}{\sqrt{2}}(1-1) = 0 \checkmark$$

3. Prove $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$

$$\vec{B} \times \vec{C} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

$$= \hat{x}(B_y C_z - B_z C_y) - \hat{y}(B_x C_z - B_z C_x) + \hat{z}(B_x C_y - B_y C_x)$$

$$\therefore \vec{A} \times (\vec{B} \times \vec{C}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_y C_z - B_z C_y & -B_x C_z + B_z C_x & B_x C_y - B_y C_x \end{vmatrix}$$

$$= \hat{x}\left[A_y\left(B_xC_y - B_yC_x\right) - A_z\left(B_zC_x - B_xC_z\right)\right]$$

$$-\hat{y}\left[A_x\left(B_xC_y - B_yC_x\right) - A_z\left(B_yC_z - B_zC_y\right)\right]$$

$$+\hat{z}\left[A_x\left(B_zC_x - B_xC_z\right) - A_y\left(B_yC_z - B_zC_y\right)\right]$$

Start by factoring out components of $\vec{B}$ & $\vec{C}$ $\hat{x}, \hat{y}, \hat{z}$ terms

$$\vec{A}\times(\vec{B}\times\vec{C}) = \hat{x}\left[B_x\left(A_yC_y + A_zC_z\right) - C_x\left(A_yB_y + A_zB_z\right)\right]$$

$$+\hat{y}\left[B_y\left(A_xC_x + A_zC_z\right) - C_y\left(A_xB_x + A_zB_z\right)\right]$$

$$+\hat{z}\left[B_z\left(A_xC_x + A_yC_y\right) - C_z\left(A_xB_x - A_yB_y\right)\right]$$

Next, add & subtract $A_xB_xC_x$, $A_yB_yC_y$, $A_zB_zC_z$

$$\therefore \vec{A}\times(\vec{B}\times\vec{C}) = \hat{x}\Big[B_x\big(\overbrace{A_xC_x + A_yC_y + A_zC_z}^{\vec{A}\cdot\vec{C}}\big) - C_x\big(\underbrace{A_xB_x + A_yB_y + A_zB_z}_{\vec{A}\cdot\vec{B}}\big)\Big]$$

$$+ \hat{y}\Big[B_y\big(\overbrace{A_xC_x + A_yC_y + A_zC_z}^{\vec{A}\cdot\vec{C}}\big) - C_y\big(\underbrace{A_xB_x + A_yB_y + A_zB_z}_{\vec{A}\cdot\vec{C}}\big)\Big]$$

$$+ \hat{z}\Big[B_z\big(\overbrace{A_xC_x + A_yC_y + A_zC_z}^{\vec{A}\cdot\vec{C}}\big) - C_z\big(\underbrace{A_xB_x + A_yB_y + A_zB_z}_{\vec{A}\cdot\vec{B}}\big)\Big]$$

$$\therefore \vec{A}\times(\vec{B}\times\vec{C}) = \hat{x}(B_x\,\vec{A}\cdot\vec{C} - C_x\,\vec{A}\cdot\vec{B}) + \hat{y}(B_y\,\vec{A}\cdot\vec{C} - C_y\,\vec{A}\cdot\vec{B}) + \hat{z}(B_z\,\vec{A}\cdot\vec{C} - C_z\,\vec{A}\cdot\vec{B})$$

$$= (B_x\hat{x} + B_y\hat{y} + B_z\hat{z})\,\vec{A}\cdot\vec{C} - (C_x\hat{x} + C_y\hat{y} + C_z\hat{z})\,\vec{A}\cdot\vec{B}$$

$$\boxed{\therefore\ \vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})}$$

4. $$\vec{E} = \frac{k_e q}{r^2}\hat{r}$$

Note that $\hat{r} = \dfrac{\vec{r}}{r}$

$$\therefore\ \vec{E} = \frac{k_e q}{r^3}\vec{r}$$

but $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

$$r^3 = (x^2+y^2+z^2)^{3/2}$$

$\therefore$ In Cartesian coord., the electric field due to a point charge is:

$$\vec{E} = \frac{k_e q}{(x^2+y^2+z^2)^{3/2}}(x\hat{x} + y\hat{y} + z\hat{z})$$

(b) $\vec{\nabla}\cdot\vec{E} = \dfrac{\partial E_x}{\partial x} + \dfrac{\partial E_y}{\partial y} + \dfrac{\partial E_z}{\partial z}$

$$\frac{\partial E_x}{\partial x} = k_e q \frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{3/2}}\right)$$

$$= k_e q \left[\frac{1}{(x^2+y^2+z^2)^{3/2}} - \frac{\frac{3}{2}x\,(2x)}{(x^2+y^2+z^2)^{5/2}}\right]$$

$$= k_e q \left[\frac{1}{r^3} - \frac{3x^2}{r^5}\right]$$

$$= \frac{k_e q}{r^3}\left[1 - 3\left(\frac{x}{r}\right)^2\right]$$

Likewise, will find

$$\frac{\partial E_y}{\partial y} = \frac{k_e q}{r^3}\left[1 - 3\left(\frac{y}{r}\right)^2\right]$$

$$\frac{\partial E_z}{\partial z} = \frac{k_e q}{r^3}\left[1 - 3\left(\frac{z}{r}\right)^2\right]$$

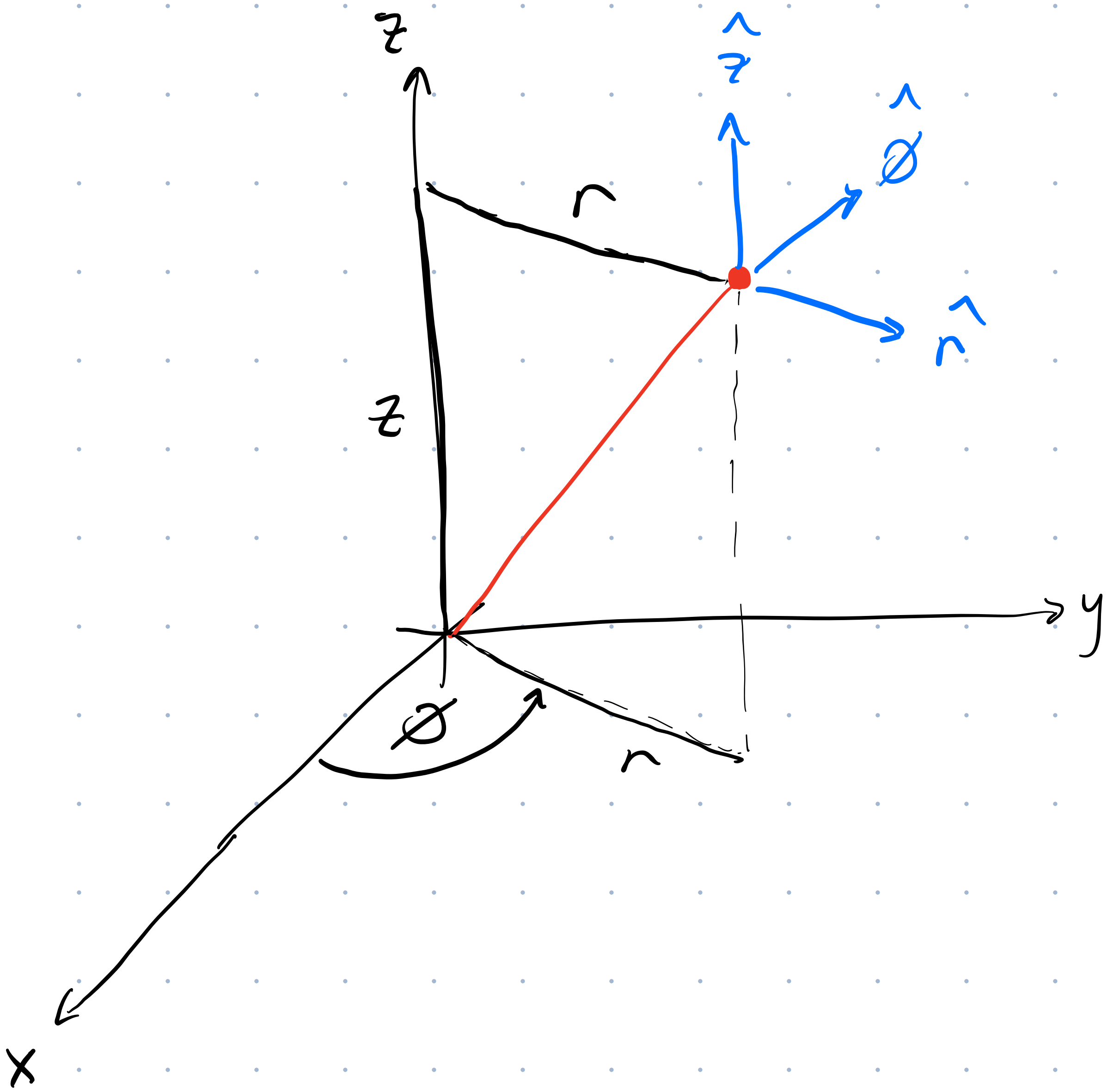
$$\therefore \vec{\nabla}\cdot\vec{E} = \frac{k_e q}{r^3}\left\{\left[1-3\left(\frac{x}{r}\right)^2\right]+\left[1-3\left(\frac{y}{r}\right)^2\right] + \left[1-3\left(\frac{z}{r}\right)^2\right]\right\}$$

$$= \frac{k_e q}{r^3}\left[3-\frac{3}{\cancel{r^2}}\cancel{\underbrace{\left(x^2+y^2+z^2\right)}_{r^2}}\right]$$

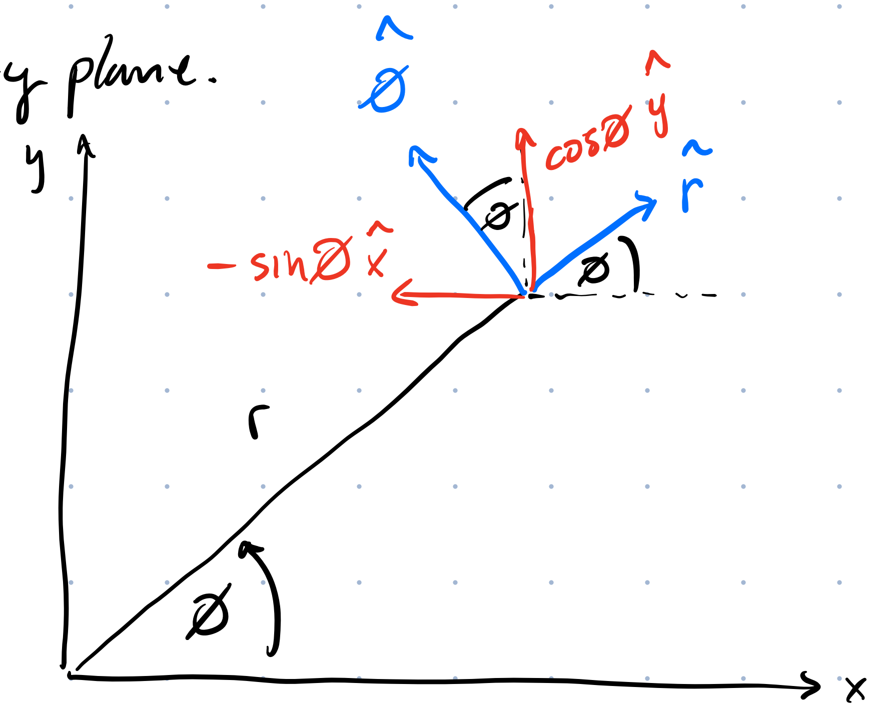
$$\boxed{\therefore \vec{\nabla}\cdot\vec{E} = 0}$$

valid everywhere except @ $r=0$ where $\vec{E}$ diverges.

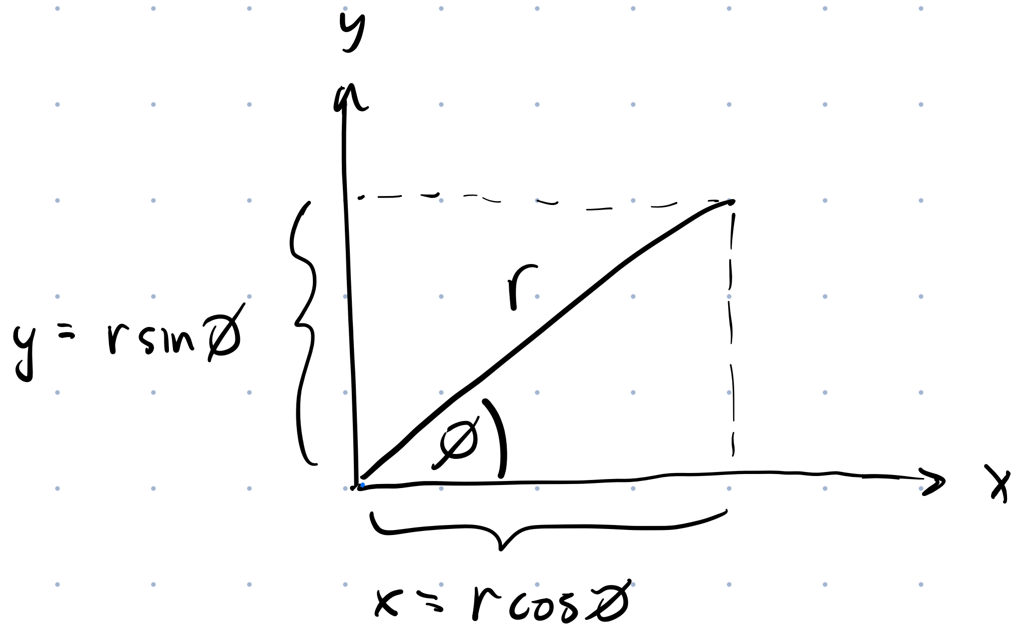
5. Cylindrical coordinates:



View from x-y plane.



From the sketch above, $\boxed{\hat{\phi} = -\sin\phi\,\hat{x} + \cos\phi\,\hat{y}}$



$$\sqrt{x^2+y^2} = r$$

$$\cos\phi = \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}}$$

$$\sin\phi = \frac{y}{r} = \frac{y}{\sqrt{x^2+y^2}}$$

$$\boxed{\therefore \hat{\phi} = \frac{-y}{\sqrt{x^2+y^2}}\hat{x} + \frac{x}{\sqrt{x^2+y^2}}\hat{y}}$$

(b) For a long straight current,

$$\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\phi} \text{ in cylindrical coords.}$$

subbing in for $\hat{\phi}$ & r from (a), we get:

$$\vec{B} = \frac{\mu_0 I}{2\pi}\frac{1}{\sqrt{x^2+y^2}}\left(\frac{-y}{\sqrt{x^2+y^2}}\hat{x} + \frac{x}{\sqrt{x^2+y^2}}\hat{y}\right)$$

$$\boxed{\therefore \vec{B} = \frac{\mu_0 I}{2\pi}\left(\frac{-y\hat{x} + x\hat{y}}{x^2+y^2}\right)}$$

(c) $$\vec{\nabla}\times\vec{B} = \frac{\mu_0 I}{2\pi}\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0 \end{vmatrix}$$

$$= \frac{\mu_0 I}{2\pi}\left[\hat{x}(0-0) - \hat{y}(0-0) + \hat{z}\left(\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)\right)\right]$$

$$= \frac{\mu_0 I}{2\pi}\hat{z}\left[\frac{1}{x^2+y^2} - \frac{x(2x)}{(x^2+y^2)^2} + \frac{1}{x^2+y^2} - \frac{y(2y)}{(x^2+y^2)^2}\right]$$

$$= \frac{\mu_0 I}{\cancel{2}\pi}\hat{z}\left[\frac{\cancel{2}}{x^2+y^2} - \frac{\cancel{2}x^2+\cancel{2}y^2}{(x^2+y^2)^2}\right]$$

$$\therefore \vec{\nabla} \times \vec{B} = \frac{\mu_0 I}{\pi} \hat{z} \left[\frac{(x^2+y^2) - (x^2+y^2)}{(x^2+y^2)^2} \right]$$

$$\boxed{\therefore \vec{\nabla} \times \vec{B} = 0}$$

True everywhere except at $r=0$ (location of the current).